Scattering theory for quantum Hall anyons in a saddle point potential

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We study the theory of scattering of two anyons in the presence of a quadratic saddle point potential and a perpendicular magnetic field. The scattering problem decouples in the center-of-mass and the relative coordinates. The scattering theory for the relative coordinate encodes the effects of anyon statistics in the two-particle scattering. This is fully characterized by two energy-dependent scattering phase shifts. We develop a method to solve this scattering problem numerically, using a generalized lowest Landau level approximation.

One of the most remarkable features of the fractional quantum Hall effect[1,2] is the existence of quasiparticles with fractional charge[3] and fractional exchange statistics.[4,5] Direct evidence for the existence of fractional charge has been observed experimentally.[6–9] One way this has been achieved is through the study of shot noise in a quantum Hall system constricted by a point contact,[7,8] by which the charge of the current carriers may be derived from the fluctuations of the current backscattered from the point contact.

Despite strong theoretical reasons to expect that these fractionally charged quasiparticles also have fractional statistics,[4,5] to date there has been no unambiguous experimental demonstration of fractional statistics. Several authors have proposed methods in which evidence of fractional statistics might appear in transport experiments,[10–14] based on mesoscopic devices of various geometries. Here we adopt a simple approach, and ask a very natural theoretical question: can anyonic statistics influence the transport through a *single* point contact? Clearly, since the effect of exchange statistics is a two-particle property, any such effect requires more than one particle to be present in the point-contact region. We therefore study the scattering properties of a pair of anyons incident on the point contact. Since the quantum Hall anyons are charged, and experience a strong magnetic field, their free motion is along edge states which follow classical equipotentials. The required scattering theory is therefore very different from that of freely moving anyons, described by a conventional kinetic energy.[15] As we show, the anyonic nature does affect the scattering of two quantum Hall quasiparticles in the point-contact region. This effect can be characterized by two energy-dependent phase shifts, which we calculate numerically for general anyonic statistics parameter.

We consider two identical particles, with mass M and charge q, subjected to a magnetic field B and in the presence of a quadratic saddle point potential

$$H = \sum_{i=1,2} \frac{1}{2M}[\boldsymbol{p}_i - q\boldsymbol{A}(\boldsymbol{r}_i)]^2 + U(y_i^2 - x_i^2), \qquad (1)$$

where $\nabla \times \boldsymbol{A} = B\hat{\boldsymbol{z}}$ is the uniform magnetic field. The particles are taken to be anyons,[16] with statistics parameter Δ. Thus,

the two-particle wave function has the boundary condition that

$$\Psi(\theta + 2\pi) = e^{-i2\pi\Delta}\Psi(\theta), \qquad (2)$$

where $\theta \rightarrow \theta + 2\pi$ represents one complete clockwise rotation of the relative coordinate of the two particles. We note that this defining feature applies either for distinguishable or indistinguishable anyons.[17] If the anyons are indistinguishable, one can impose the stronger restriction

$$\Psi(\theta + \pi) = e^{-i\pi\Delta}\Psi(\theta) \quad \text{[indistinguishable].} \qquad (3)$$

Then, one recovers the exchange statistics for bosons and fermions for $\Delta = 0$ and 1, respectively. For greater generality, we study the case of distinguishable anyons. The effects of the additional restriction for indistinguishability (3) will be made clear in the discussion.

We simplify the problem, defined by Eq. (1) and the boundary condition (2), by introducing a "center-of-mass" coordinate, $\boldsymbol{r}_c$, and a "relative" coordinate, $\boldsymbol{r}_r$, with

$$\boldsymbol{r}_c \equiv \frac{1}{\sqrt{2}}(\boldsymbol{r}_1 + \boldsymbol{r}_2) \quad \boldsymbol{r}_r \equiv \frac{1}{\sqrt{2}}(\boldsymbol{r}_1 - \boldsymbol{r}_2). \qquad (4)$$

The Hamiltonian separates, becoming

$$H_\alpha \equiv \sum_{\mu=c,r} \frac{1}{2M}[\boldsymbol{p}_\mu - q\boldsymbol{A}(\boldsymbol{r}_\mu)]^2 + U(y_\mu^2 - x_\mu^2). \qquad (5)$$

Thus, the two-particle problem can be expressed as two independent one-particle problems, and the total energy $E = E_c + E_r$ can be divided into separately conserved contributions from the center-of-mass and relative coordinates.

The center-of-mass coordinate $\boldsymbol{r}_c$ is insensitive to anyonic statistics. The scattering theory for this coordinate is identical to the one-particle problem solved by Fertig and Halperin.[18] This work allows one to deduce the transmission and reflection coefficients of the center-of-mass coordinate moving in the saddle point potential, in terms of E_c.

The relative coordinate $\boldsymbol{r}_r$ has the same Hamiltonian. However, since the wave function has the additional anyonic boundary condition (2), the results of Ref. 18 cannot be applied. The solution of the scattering problem for the relative coordinate is the central result of the present paper. From here on, for simplicity, we drop the subscripts r on $\boldsymbol{r}_r$ and E_r,

with the understanding that the calculation refers only to the relative coordinate.

Before considering the scattering problem, we study first the spectrum for the relative coordinate in the absence of a potential, $U=0$. For $\Delta=0$, this leads to the familiar Landau level states and spectrum. The anyon boundary condition changes the nature of these states. To describe the generalized Landau level states, we use polar coordinates, $\mathbf{r} = r(\sin\theta, \cos\theta)$ and the symmetric gauge $\mathbf{A} = -\mathbf{e}_\theta r B / 2$. The eigenstates are

$$\psi_{n,m}(r,\theta) = e^{-i(m+\Delta)\theta} R_{n,m}(r), \qquad (6)$$

where m is an integer, and $n=0,1,2,\ldots$. The normalized radial wave functions are

$$R_{n,m}(r) = \mathcal{N}\frac{1}{\ell}\left(\frac{r}{\ell}\right)^{|m+\Delta|} L_n^{|m+\Delta|}\left(\frac{r^2}{2\ell^2}\right) e^{-r^2/4\ell^2}, \qquad (7)$$

where $\ell \equiv \sqrt{\hbar/(qB)}$ is the magnetic length, L_n^m are the Laguerre polynomials, and the normalization is $\mathcal{N} \equiv \sqrt{n!/(\Gamma(|m+\Delta|+n+1)2\pi 2^{|m+\Delta|})}$. The energy is

$$E_{n,m} = \hbar\omega_c\left[n + \frac{1}{2}|m+\Delta| - \frac{1}{2}(m+\Delta) + \frac{1}{2}\right], \qquad (8)$$

where $\omega_c \equiv qB/M$ is the cyclotron frequency. The spectrum has the required feature that it is invariant under the transformation $\Delta \to \Delta' = \Delta - 1$, equivalent to the insertion of a flux quantum at the origin. This amounts to the change $m \to m' = m+1$. Thus, it is sufficient to study the range $0 \le \Delta < 1$ to cover all possible cases. For positive $m+\Delta \ge 0$ the energy depends only on n: these sets of states thus form highly degenerate Landau levels for the relative motion of the anyons.

We now reintroduce the potential $U \ne 0$. We do this within a (generalized) lowest Landau level approximation, in which U is taken only to lift the degeneracy of the lowest energy states. This corresponds to retaining only those levels with $n=0$ and $m+\Delta \ge 0$. Noting that the lowest energy state lies higher in energy by $\hbar\omega_c(1-\Delta)$, the lowest Landau level approximation for the relative motion of the anyons is valid for $U\ell^2 \ll \hbar\omega_c(1-\Delta)$. We denote the set of degenerate basis states as $|j\rangle$, where $j=0,1,2,\ldots$, and expand the wave function as $|\psi\rangle \equiv \Sigma \psi_j|j\rangle$. The Schrödinger equation becomes

$$\epsilon\psi_j = \sqrt{(j+\Delta-1)(j+\Delta)}\,\psi_{j-2} + \sqrt{(j+\Delta+1)(j+\Delta+2)}\,\psi_{j+2} \qquad (9)$$

for the amplitudes ψ_j, where $\epsilon \equiv E/(U\ell^2)$ is the dimensionless measure of the energy. Note that there are no terms that couple odd and even values of j. Physically, this arises from

the fact that the potential is invariant under spatial inversion $(x,y) \to (-x,-y)$, and so the parity of the wave function is a good quantum number. Thus the Schrödinger equation takes the form of two decoupled sets of difference equations. We write the general solution in terms of the "even" and "odd" channels as

$$|\psi^e\rangle = \sum_{p=0}^{\infty} \psi_{2p}^e |2p\rangle \qquad (10)$$

$$|\psi^o\rangle = \sum_{p=0}^{\infty} \psi_{2p+1}^o |2p+1\rangle. \qquad (11)$$

For distinguishable anyons, the wave function can be a linear superposition of these two solutions. However, for indistinguishable anyons, the boundary condition (3) requires that only the even solution contributes.

We shall construct the wave function at large distances from the origin, $r \gg \ell$. In this limit, contributions are from single-particle states with $j \gg 1$. For $j \gg \epsilon, 1$, the Schrödinger equation (9) has the wavelike solutions (normalized to unit density per orbital j)[19]

$$\psi_j = e^{i\theta_0 j} \quad [\theta_0 = \pm \pi/4, \pm 3\pi/4]. \qquad (12)$$

These solutions can be viewed as incoming and outgoing waves in the discrete semi-infinite one-dimensional system defined by the sites $j=0,1,2,\ldots$. To understand the nature of these solutions, it is useful to construct their spatial wave functions

$$\langle \mathbf{r}|\psi\rangle = \sum_j e^{i\theta_0 j} e^{-i(j+\Delta)\theta} R_{0,j}(r). \qquad (13)$$

At large radius r, the wave function has significant amplitude under the condition that $\theta \simeq \theta_0 = \pm\pi/4, \pm 3\pi/4$. That is, the wave function is peaked in these four angular directions. These are the directions along which the zero-energy equipotentials of the electrostatic potential extend. Recalling that, in the semiclassical approximation, the particle moves along the equipotentials of the electrostatic potential, one sees that the derived angles correspond to the two incoming $(-\pi/4, 3\pi/4)$ and the two outgoing $(\pi/4, -3\pi/4)$ channels of the saddle point potential.

For a general scattering problem on a semi-infinite one-dimensional system (e.g., waves on a string which is clamped at one end), at a fixed energy (frequency) one expects there to be only *two* wavelike solutions at large distances; these can be taken to be the incoming and the outgoing waves. That, in the present case, there are *four* wavelike solutions, is a special feature of the problem, which arises from the fact that (as above) the Schrödinger equation (9) conserves the parity. That is, the sites with j even and the sites with j odd each behave as independent semi-infinite one-dimensional systems. For each parity (even or odd) there is one incoming mode and one outgoing mode. It is convenient to re-express the four modes [Eq. (12)] in terms of modes of definite parity. The explicit forms (normalized to unit density per orbital j) are

$$\psi_j^{\text{e/o,in}} = \frac{1}{\sqrt{2}}[e^{-i\pi/4j} \pm e^{i3\pi/4j}] \tag{14}$$

$$\psi_j^{\text{e/o,out}} = \frac{1}{\sqrt{2}}[e^{i\pi/4j} \pm e^{-i3\pi/4j}], \tag{15}$$

which are readily verified to have the feature that ψ_j^{e} (ψ_j^{o}) is nonzero only for $j=$even (odd). The "in" and "out" labels are identified by the fact that the states have large amplitude on the incoming ($\theta=-\pi/4, 3\pi/4$) or outgoing ($\theta=\pi/4, -3\pi/4$) channels of the saddle point potential.

For the semi-infinite one-dimensional scattering problem, the asymptotic (large distance) incoming and outgoing waves are coupled, with scattering from incoming to outgoing waves occurring at small distances. In the problem of interest here this scattering at small distances (small j) conserves parity, so the incoming mode in the even (odd) channel can scatter only to the outgoing mode in the even (odd) channel. By conservation of particle flux, the scattering from incoming to outgoing modes can amount only to a phase shift. Hence, the (unnormalized) energy eigenstates are of the form

$$\psi_j^{\text{e/o}} \sim \psi_j^{\text{e/o,in}} + e^{i\zeta^{\text{e/o}}(\epsilon)}\psi_j^{\text{e/o,out}} \quad [j \gg \epsilon, 1]. \tag{16}$$

Thus, the energy eigenstates are fully characterized by two scattering phases: $\zeta^{\text{e}}(\epsilon)$ and $\zeta^{\text{o}}(\epsilon)$ which correspond to the even- and odd-parity wave functions.

For indistinguishable anyons [Eq. (3)] only ψ^{e} is relevant. There is only one incoming and one outgoing channel for the relative coordinate, so the scattering is described only by a single phase shift, $\zeta^{\text{e}}(\epsilon)$.

For distinguishable particles, both ψ^{e} and ψ^{o} can contribute. In this case, it is instructive to disentangle the above transformation into states of definite parity, and to determine the *transmission probability*. This is defined as the probability for transmission from a state that is an incoming wave along the definite angular direction $\theta \simeq -\pi/4$, into a state that is outgoing along $\theta \simeq +\pi/4$. (These angles match the convention chosen in Ref. 18.) The incoming wave (normalized to unit density per orbital) is

$$e^{-i\pi/4j} = \frac{1}{\sqrt{2}}[\psi_j^{\text{e,in}} + \psi_j^{\text{o,in}}], \tag{17}$$

which is scattered into the state

$$\frac{1}{\sqrt{2}}[e^{i\zeta^{\text{e}}}\psi_j^{\text{e,out}} + e^{i\zeta^{\text{o}}}\psi_j^{\text{o,out}}]. \tag{18}$$

Noting that the transmitted wave is

$$e^{+i\pi/4j} = \frac{1}{\sqrt{2}}[\psi_j^{\text{e,out}} + \psi_j^{\text{o,out}}] \tag{19}$$

and using the fact that $\psi^{\text{e,out}}$ and $\psi^{\text{o,out}}$ are orthogonal, one sees that the transmission probability is

$$T = \left|\frac{1}{2}(e^{i\zeta^{\text{e}}} + e^{i\zeta^{\text{o}}})\right|^2 = \frac{1}{2}\{1 + \cos[\zeta^{\text{e}}(\epsilon) - \zeta^{\text{o}}(\epsilon)]\}. \tag{20}$$

We have determined the functions $\zeta^{\text{e/o}}(\epsilon)$—thereby solving the scattering problem for the relative coordinate—by a numerical construction of the Green's function of the discrete Hamiltonian in Eq. (9). We study an approximate version of the full model, in which we treat $j=0, N$ according to the exact Hamiltonian (9), but take the hopping matrix elements for $j=N+1, \infty$ to be constant, and equal to $\sqrt{(N+\Delta-1)(N+\Delta)}$. The method becomes increasingly accurate as $N \to \infty$. Following standard techniques,[20] the region with $j \geq N+1$ can be replaced by a self-energy, and the Green's function for $j \leq N$ is

$$\hat{G}(\epsilon) = [\epsilon\hat{I} - \hat{H} - \hat{\sigma}]^{-1}, \tag{21}$$

where the self-energy has matrix representation

$$\sigma_{ij} = \frac{1}{2}\delta_{i,N}\delta_{j,N}[\epsilon - i\sqrt{4(N+\Delta+1)(N+\Delta+2) - \epsilon^2}]. \tag{22}$$

The Green's function for $j \leq N$ is then found by numerical inversion of the finite matrix (for large finite N).

Using the fact that, for a given element i, the Green's function $G_{i,j}$ for $j < i$ is an energy eigenfunction, we can use this to compare to the wave function [Eq. (16)] in the asymptotic regime $j \to \infty$. Using Eq. (16), the scattering phase can be extracted from the ratios

$$\frac{\psi_{4p+2}^{\text{e}}}{\psi_{4p}^{\text{e}}} \xrightarrow{p \to \infty} \tan\left[\frac{-\zeta^{\text{e}}(\epsilon) + \frac{\epsilon}{2}\ln(4p)}{2}\right] \tag{23}$$

$$\frac{\psi_{4p+3}^{\text{o}}}{\psi_{4p+1}^{\text{o}}} \xrightarrow{p \to \infty} \tan\left[\frac{-\zeta^{\text{o}}(\epsilon) - \pi/2 + \frac{\epsilon}{2}\ln(4p)}{2}\right], \tag{24}$$

where the logarithms on the right-hand side follow from the corrections to the wave functions (12) described in Ref. 19. In this way we can numerically construct the scattering phases $\zeta^{\text{e/o}}(\epsilon)$ and from these the transmission probability T [Eq. (20)]. The numerical results converge rapidly with increasing N, becoming independent of system size for $N \gtrsim 100$. We show results for $N=2000$.

Figures 1 and 2 show our numerical results for the scattering phase shifts for the even- and odd-parity channels, respectively, as a function of energy for several values of Δ. These two functions, over the range $0 \leq \Delta < 1$, fully describe the scattering properties of the relative coordinate of quantum Hall anyons in the lowest Landau level. The results shown for $\Delta=1$ correspond to the case $\Delta=1-$, in which the state $m=-1$ remains excluded from contribution to the lowest Landau level. In this case, the spectrum is identical to that for $\Delta'=0$ and $m'=m+1$, but with the removal of the state at $m'=0$. From Eq. (16), and taking account of a $\pi/2$ phase shift arising from the change $m'=m+1$, one expects $\zeta_{\Delta=1-}^{\text{e}}(\epsilon) = \zeta_{\Delta'=0}^{\text{o}}(\epsilon) + \pi/2$, which is indeed found to hold to high accuracy in the numerical results.

In Fig. 3 we have used these results to determine the transmission coefficient [Eq. (20)]. For $\Delta=0$, the results accurately reproduce the exact analytical solution,[18] showing that our method is working correctly and is well converged.
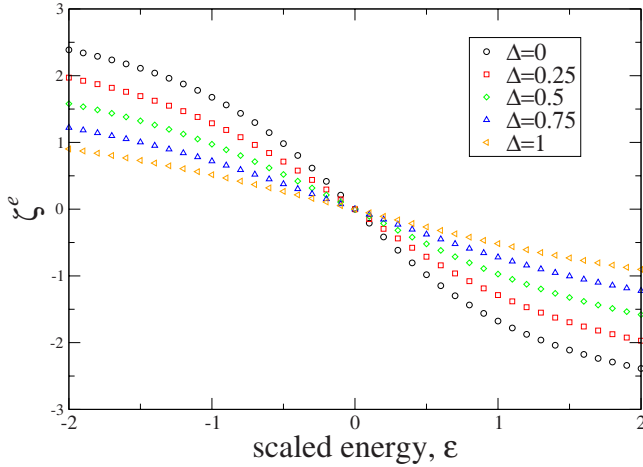
FIG. 1. (Color online) Results of numerical calculations of the scattering phase shift for the relative motion of two anyons in the even-parity channel, $\zeta^e(\epsilon)$, as a function of dimensionless energy $\epsilon = E/U\ell^2$ at several values of the anyonic statistics parameter Δ.

For $\Delta \neq 0$, the results of Ref. 18 do not apply, and no analytic solution is available. As compared to the case $\Delta = 0$, the effect of increasing Δ is a broadening of the width in energy over which the transmission coefficient rises from 0 ($\epsilon \ll -1$) to 1 ($\epsilon \gg 1$). Thus, our results show that increasing Δ leads to an *increase* in the tunneling rate through the saddle point. Indeed, we find that, to a very good approximation, the results can be fitted by the function

$$T(\epsilon) = \frac{1}{1 + \exp\left(-\dfrac{\pi\epsilon}{\alpha}\right)} \qquad (25)$$

with the fit parameter α given approximately by $\alpha = 1 + 1.55378\Delta + 0.277179\Delta^2$. (See the inset to Fig. 3.)

We note that our results apply for the case of two anyons in a symmetrical saddle point potential [Eq. (1)]. The more general case can be considered by noting that $U_y y^2 - U_x x^2 = \frac{1}{2}(U_y + U_x)(y^2 - x^2) + \frac{1}{2}(U_y - U_x)r^2$, leading to an additional central (rotationally invariant) term $\propto r^2$. This term modifies
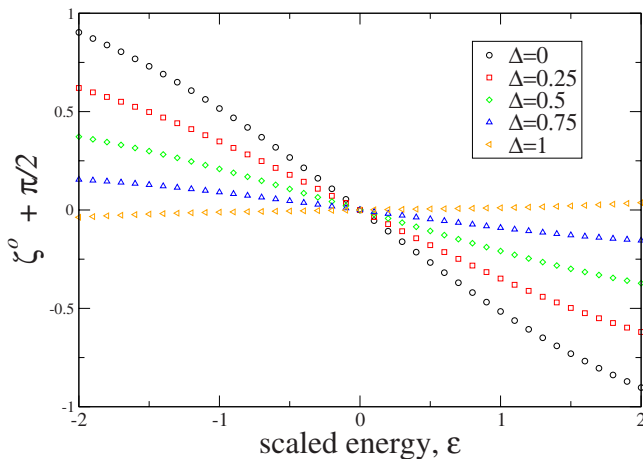

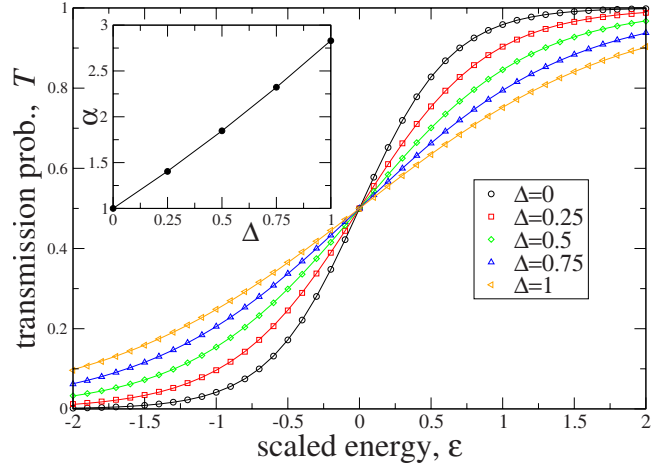FIG. 2. (Color online) Same as Fig. 1, but for the odd-parity channel, $\zeta^o(\epsilon)$.

FIG. 3. (Color online) Transmission probability T for the relative coordinate of two quantum Hall anyons, as a function of the scaled energy $\epsilon \equiv E_r/(U\ell^2)$. The points are results of numerical calculations for several values of the anyon statistics parameter Δ. The lines are fits, using the function (25) with parameter α chosen as shown in the inset. For $\Delta = 0$ choosing $\alpha = 1$ gives the exact analytic result (Ref. 18); in this case, the comparison with numerical results illustrates the accuracy of our numerical approach at this system size ($N = 2000$).

the Schrödinger equation (9) and could lead to a change in the phase shifts and transmission probabilities. In the same way, central (rotationally invariant) anyon-anyon interactions could be included within the same formalism. The solution of these more general cases is beyond the scope of the present paper, so the influence of these perturbations on the scattering properties remains an open question.

In summary, we have provided a solution of the scattering problem for two anyons in a quadratic saddle point potential and perpendicular magnetic field, through separation into center-of-mass and relative coordinates. The scattering theory for the center-of-mass coordinate has previously been solved analytically.[18] The scattering for the relative coordinate is characterized by two energy-dependent phase shifts (for even and odd parities). We have computed these phase shifts within a lowest Landau level approximation. Our results provide a complete solution of the two-anyon scattering problem. They show that the two-particle scattering properties in the vicinity of a point contact depend on the anyonic statistics parameter. This shows that one can hope to obtain experimental signatures of anyonic statistics in point-contact devices. We believe that the approach and solution we have described provide a useful basis on which to build further theoretical studies of nonequilibrium properties of anyons—under conditions of bias and/or "beam dilution[21]" where multiple quasiparticles may enter the point-contact region. Depending on the experimental setup, and the relevant observables, the two-particle scattering problem in the coordinates r_1 and r_2 should be decomposed into relative and center-of-mass coordinates for which the scattering properties follow the results presented.

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